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DERIVATION, STABILTY ANALYSIS AND IMPLEMENTATION OF SECOND DERIVATIVE LINEAR MULTISTEP METHODS FOR STIFF INITIAL VALUE PROBLEM

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A B S T R A C T

This paper presents the blended method for the solution of stiff initial value problems. It's developed using the Backward Differentiation Formula, Adams Moulton Method for step size k=2, 3 & 4. The derivation is achieved using the multistep collocation approach with matrix inversion techniques. From the continuous formulations, corresponding discrete schemes are generated. A convergence study is also conducted on the discrete block schemes and all are shown to be consistent and zero stable. The plots of the regions of absolute stability shows that Adams Moulton Backward Differential Formula k=3, 4 are A-stable while Adams Moulton Backward Differential Formula k=2 is A(α) stable. The blended variants demonstrated higher accuracy compared to the second derivative Backward Differentiation Formula and second derivative Adams Moulton Method when applied in block form to solve some stiff problems.

1. Introduction

Stiff initial value problems (IVP) in ordinary differential equations (ODEs) are ubiquitous in modelling physical, chemical, and biological systems where solutions exhibit multiple time-scales. In these systems, standard explicit methods often require prohibitively small-time steps for stability, making implicit or specially tailored methods necessary. (Amurawaye, *et al*., 2023), worked on the solution of stiff initial value problem (IVP). Some problems cannot be solved exactly, so it is common to seek approximate solutions by means of numerical methods. Nowadays with numerical analysis, we can achieve a high accuracy with a reliable bound on error between the analytical solution and its numerical approximation at a very low cost. Stiff systems are considered difficult because numerical methods designed for non-stiff problems are used with very small step sizes or not converge at all. However, it is not all differential equations which arise from the modeling of physical phenomena that possess analytic (exact) solutions. Hence the development of numerical (computational) methods to obtain approximate solutions becomes appropriate. For the numerical integration of ODEs. They are linear multistep methods that, for a given function and time approximate the derivative of that function using information from already computed times, thereby increasing the accuracy of the approximation.

Implicit methods have been a key focus for solving stiff equations because they offer enhanced stability compared to explicit methods. Backward differentiation formulas (BDF) and Adams Moulton Methods (AMM) remain among the most popular techniques due to their ability to handle stiff problems efficiently. BDF methods are widely employed in solving stiff ODEs, particularly for applications in chemical kinetics and circuit simulation. The primary benefit of BDF is its A-stability, allowing the method to handle stiff problems without an excessively small-time step. Kumar and Rathod (2022) applied BDF methods to recurrent neural network. Brennan and Jacobson (2021) addressed the limitation by developing a modified BDF. Chen *et al.* (2017) proposed an adaptive version of BDF with enhanced stability properties, extending the method to high-order stiff problems. Numerical methods for solving stiff initial value problems (IVPs) are of the form:

$$y' = f(x, y),\ \ y(x_0) = y_0\ \ x \in [a, b] \qquad (1)$$

One approach to enhance the performance of Linear Multistep Methods (LMMs) is the inclusion of higher derivative terms, most notably second derivatives in the formulation. Second derivative linear multistep methods leverage an extra derivative term to gain additional degrees of freedom in design, which can be exploited to improve stability while maintaining or increasing the order of accuracy. Solomon & Sofoluwe (2020) generalized second derivative LMM based on the methods of Enright. Other researchers who worked on higher derivatives include: Kumleng G.M *et al*. (2015), Okuonghae, & Ikhile, (2015), Ali *et al.* (2015), Yakubu & Markus (2016), Sunday *et al*. (2015).

Adaptive strategies have also been considered. Ajewole *et al*. (2024), introduces parameterized families of second derivative backward differentiation formulas (and their extensions), adjusting adaptively to stiffness', the stability properties are shown to be

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improved over non-adaptive versions, and numerical experiments verify their effectiveness in chemical reaction models. In this paper, we develop and apply second derivative linear multistep methods that can effectively solve stiff initial value problems while addressing issues of stability, computational cost, and accuracy, by incorporating the Adams Moulton Method (AMM) and Backward differentiation formulae (BDF) through the use of multistep collocation method by matrix inversion technique.

2. Methods

In this section, we shall consider the derivation of the continuous formulations of the method. Its continuous forms shall be obtained through multistep collocation method using the matrix inversion technique and we deduced the discrete schemes from it. The order, error constants, zero stability, consistency and convergence of the discrete schemes and their regions of absolute stability were studied and plotted.

Derivation of Multistep Collocation Method

A k-step multistep collocation method with m collocation points was obtained as

$$Y(x) = \sum_{j=0}^{t-1} \alpha_j(x) y_{n+j} + h \sum_{j=o}^{m-1} \beta_j(x) f_{n+j} \qquad (2)$$

We define $\alpha_j(x)$ and $\beta_j(x)$ are continuous coefficients of the method defined as

$$\alpha_j(x) = \sum_{i=0}^{t+m-1} \alpha_{j,i+1} x^i \qquad j = \{0,1,\dots,t-1\} \qquad (3)$$

$$\beta_j(x) = \sum_{i=0}^{t+m-1} h\beta_{j,i+1} x^i \qquad j = \{0,1,\dots,m-1\} \qquad (4)$$

Where $x_0, \dots, x_{m-1}$ are the m collocation points and $x_{n+j}, j = 0,1,2,\dots,t-1$ are the arbitrarily chosen interpolation points.
To get $\alpha_j(x)$ and $\beta_j(x)$, as in R. O. Akinola and K. J. Ajibade (2021) arrived at a matrix equation of the form

$$DC = I \qquad (5)$$

Where I the identity matrix of dimension is $(t+m) \times (t+m)$, while D and C are matrices defined by

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \dots & x_{n+1}^{t+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2x_0 & \dots & (t+m-1)x_0^{t+m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{m-1} & \dots & (t+m-1)x_{m-1}^{t+m-2} \end{bmatrix} \qquad (6)$$

$$C = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & \dots & \alpha_{t-1,1} & h\beta_{0,1} & \cdots & h\beta_{m-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{t-1,2} & h\beta_{0,2} & \cdots & h\beta_{m-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,t+m} & \alpha_{1,t+m} & \cdots & \alpha_{t-1,t+m} & h\beta_{0,t+m} & \dots & h\beta_{m-1,t+m} \end{bmatrix} \qquad (7)$$

It follows from (5) that the columns of $C = D^{-1}$ give the continuous coefficients of the continuous scheme (2).
Subsequently, the continuous formulations of the method shall be derived.

Derivation of Continuous Formulation of Adams Moulton Backward Differential Formula k=2

The number of interpolation points t =1 and the number of collocation points m =4, Therefore (2) becomes:

$$y(x) = \alpha_1(x) y_{n+1} + \mathrm{h}[\beta_0 f_n + \beta_1(x) f_{n+1} + \beta_2(x) f_{n+2}] + h^2 \gamma_2(x) g_{n+2} \qquad (8)$$

$$D = \begin{bmatrix} 1 & x_n + h & (x_n+h)^2 & (x_n+h)^3 & (x_n+h)^4 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 \\ 0 & 1 & 2x_n + 2h & 3(x_n+h)^2 & 4(x_n+h)^3 \\ 0 & 1 & 2x_n + 4h & 3(x_n+2h)^2 & 4(x_n+2h)^3 \\ 0 & 0 & 2 & 6x_n + 12h & 12(x_n+2h)^2 \end{bmatrix} \qquad (9)$$

The inverse of the matrix $C = (D^{-1})$ is computed using Maple16 from which the continuous scheme is obtained and evaluating it at $x = x_{n,}$ and $x_{n+2,}$ the following discrete schemes are obtained

$$y_{n+2} = -\frac{1}{8} h^2 g_{n+2} + \frac{1}{48}(-f_n + 20f_{n+1} + 29f_{n+2})h + y_{n+1}$$
$$y_{n+1} = -\frac{1}{8} h^2 g_{n+2} + \frac{1}{48}(-17f_n - 44f_{n+1} + 13f_{n+2})h + y_n$$
(10)

Derivation of Continuous Formulation of Adams Moulton Backward Differential Formula k=3

The number of interpolation points t =1 and the number of collocation points m =5, Therefore (2) becomes:

$$y(x) = \alpha_1(x)y_{n+1} + h[\beta_0 f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3}] + h^2\gamma_3(x)g_{n+3} \quad (11)$$

$$D = \begin{bmatrix} 1 & x_n + 2h & (x_n+2h)^2 & \cdots & (x_n+2h)^5 \\ 0 & 1 & 2x_n & \cdots & 5x_n^4 \\ 0 & 1 & 2x_n+2h & \cdots & 5(x_n+h)^4 \\ 0 & 1 & 2x_n+4h & \cdots & 5(x_n+2h)^4 \\ 0 & 1 & 2x_n+6h & \cdots & 5(x_n+3h)^4 \\ 0 & 0 & 2 & \cdots & 20(x_n+3h)^3 \end{bmatrix} \quad (12)$$

The inverse of the matrix $C = (D^{-1})$ is computed using Maple16 from which the continuous scheme is obtained and evaluating it at $x = x_{n,}\ x_{n+1}$, and $x_{n+3,}$ the following discrete schemes are obtained

$$y_{n+3} = -\frac{19}{180}h^2 g_{n+3} + \frac{1}{1080}(7f_n - 54f_{n+1} + 513f_{n+2} + 614f_{n+3})h + y_{n+2}$$

$$y_{n+1} = -\frac{11}{180}h^2 g_{n+3} - \frac{1}{1080}(23f_n - 486f_{n+1} - 783f_{n+2} + 166f_{n+3})h + y_{n+2}$$

$$y_n = \frac{2}{45}h^2 g_{n+3+}\frac{1}{135}(-43f_n - 189f_{n+1} - 27f_{n+2} - 11f_{n+3})h + y_{n+2} \quad (13)$$

Derivation of Continuous Formulation of Adams Moulton Backward Differential Formula k=4

The number of interpolation points t =1 and the number of collocation points m =6, Therefore (2) becomes:

$$y(x) = \alpha_1(x)y_{n+1} + h[\beta_0 f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3} + \beta_4(x)f_{n+4}] + h^2\gamma_4(x)g_{n+4} \quad (14)$$

$$D = \begin{bmatrix} 1 & x_n + 2h & (x_n+2h)^2 & (x_n+2h)^3 & \cdots & (x_n+2h)^6 \\ 0 & 1 & 2x_n & 3x_n^2 & \cdots & 6x_n^4 \\ 0 & 1 & 2x_n+2h & 3(x_n+h)^2 & \cdots & 6(x_n+h)^5 \\ 0 & 1 & 2x_n+4h & 3(x_n+2h)^2 & \cdots & 6(x_n+2h)^5 \\ 0 & 1 & 2x_n+6h & 3(x_n+3h)^2 & \cdots & 6(x_n+3h)^5 \\ 0 & 1 & 2x_n+8h & 3(x_n+4h)^2 & \cdots & 6(x_n+4h)^5 \\ 0 & 0 & 2 & 6x_n+24h & \ldots & 30(x_n+4h)^4 \end{bmatrix} \quad (15)$$

The inverse of the matrix $C = (D^{-1})$ is computed using Maple16 from which the continuous scheme is obtained and evaluating it at $x = x_{n,}\ x_{n+1}, x_{n+3}$and $x_{n+4,}$ the following discrete schemes are obtained

$$y_{n+4} = -\frac{1}{18}h^2 g_{n+4} + \frac{1}{1080}(3f_n - 32f_{n+1} + 468f_{n+2} + 1248f_{n+3} + 473f_{n+4})h + y_{n+2}$$

$$y_{n+3} = \frac{11}{288}h^2 g_{n+4} + \frac{1}{17280}(99f_n - 896f_{n+1} + 8964f_{n+2} + 10944f_{n+3} - 1831f_{n+4})h + y_{n+2}$$

$$y_{n+1} = \frac{11}{288}h^2 g_{n+4} + \frac{1}{17280}(291f_n - 7424f_{n+1} - 12924f_{n+2} + 4416f_{n+3} - 16f_{n+4})h + y_{n+2}$$

$$y_n = -\frac{1}{18}h^2 g_{n+4} + \frac{1}{1080}(-333f_n - 1568f_{n+1} - 108f_{n+2} - 288f_{n+3} + 137f_{n+4})h + y_{n+2} \quad (16)$$

Order and Error Constant

Here the order and error constants of the discrete schemes in (10) are found in block form as follows:

$$c_0 = \alpha_0 + \alpha_1 + \alpha_2 = \begin{bmatrix}0\\0\end{bmatrix}$$

$$c_1 = \alpha_1 + 2\alpha_2 - [\beta_0 + \beta_1 + \beta_2] = \begin{bmatrix}0\\0\end{bmatrix}$$

$$c_2 = \frac{1}{2!}[\alpha_1 + 2^2\alpha_2] - [\beta_1 + 2\beta_2] = \begin{bmatrix}0\\0\end{bmatrix}$$

$$c_3 = \frac{1}{3!}[\alpha_1 + 2^3\alpha_2] - \frac{1}{2!}[\beta_1 + 2\beta_2] = \begin{bmatrix}0\\0\end{bmatrix}$$

$$c_4 = \frac{1}{4!}[\alpha_1 + 2^4\alpha_2] - \frac{1}{2!}[\beta_1 + 2\beta_2] = \begin{bmatrix}1/8\\-1/15\end{bmatrix}$$

Therefore, (10) has order $p = 3$, and error constants = $\frac{1}{8}, -\frac{1}{15}$

Similarly, the other discrete schemes follow the same approach

The discrete scheme in (13) has order $p = 5$, and error constant= $\frac{97}{7200}, \frac{2}{225}, \frac{9}{800}$

The discrete scheme in (16) has order $p = 6$, and error constant c_7=$-\frac{1}{126}, -\frac{97}{30240}, -\frac{19}{10080}, -\frac{1}{1890}$

Consistency
All the schemes derived have their order greater than one. Hence the schemes are consistent.

Zero Stability
The zero stability of the discrete schemes in (10) is determined in block form as follows. Its characteristics polynomial is given as

$$\tau = \lambda A_1 - A_0 \tag{17}$$

where

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

Substituting A_0 and A_1 into (17) yields

$$\tau = \begin{bmatrix} -\frac{8}{7}\lambda & \lambda + \frac{1}{7} \\ \lambda & -1 \end{bmatrix} \tag{18}$$

Finding the determinant of (18) and solving, we get $\lambda = 0,1$
This shows that it satisfies the root condition, hence it is zero stable.

Similarly, the other discrete schemes follow the same approach
The discrete scheme in (13) is determined as $\lambda = 1,0,0$
The discrete scheme in (16) is determined as $\lambda = 1,0,0,0$

Convergence
The block discrete schemes methods are convergent since they are both consistent and zero-stable.

Stability Analysis

In this section, we shall plot the regions of absolute stability of the block methods. This is achieved by differentiating the characteristics polynomials of each block methods with respect to z which yields results. These results as well as the characteristic polynomials for each block methods are substituted into MATLAB to produce the required region of absolute stability shown in figure (1), (2), (3), (4), (5), (6), (7), (8), and (9) below;

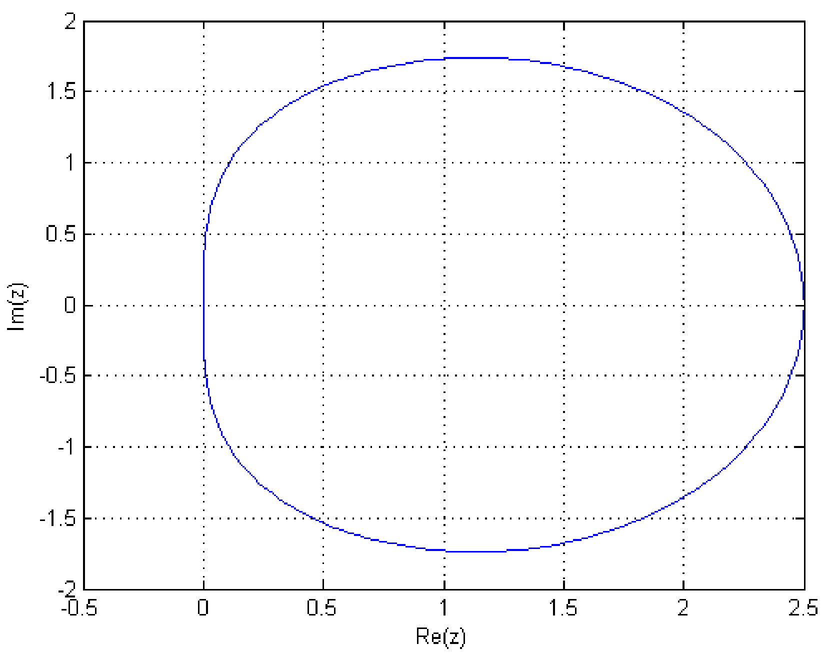

Figure 7: the region of absolute stability of the block Adams Moulton Backward Differential Formula for step number k=2

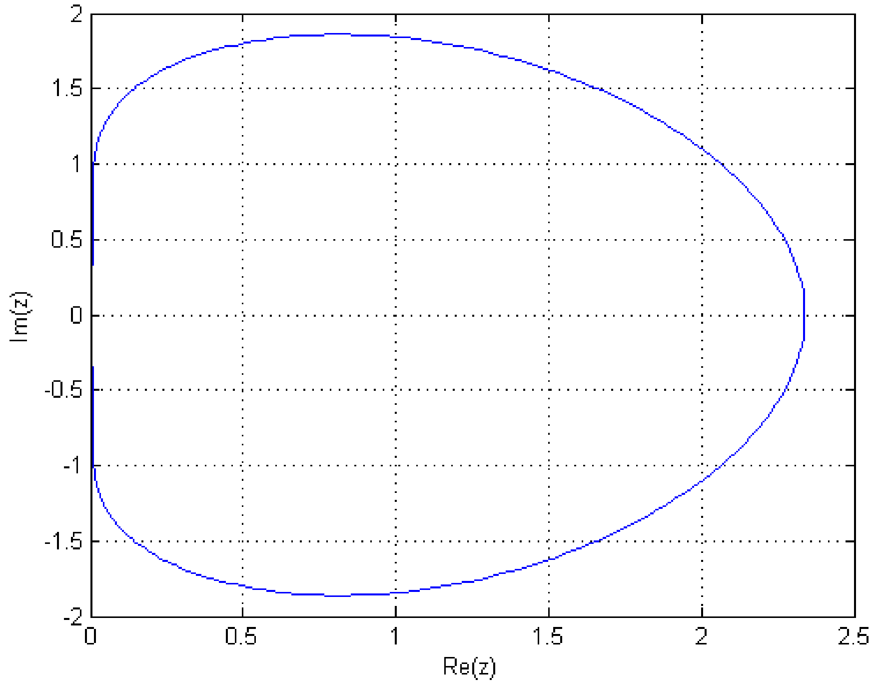

Figure 8: the region of absolute stability of the block Adams Moulton Backward Differential Formula for step number k=3

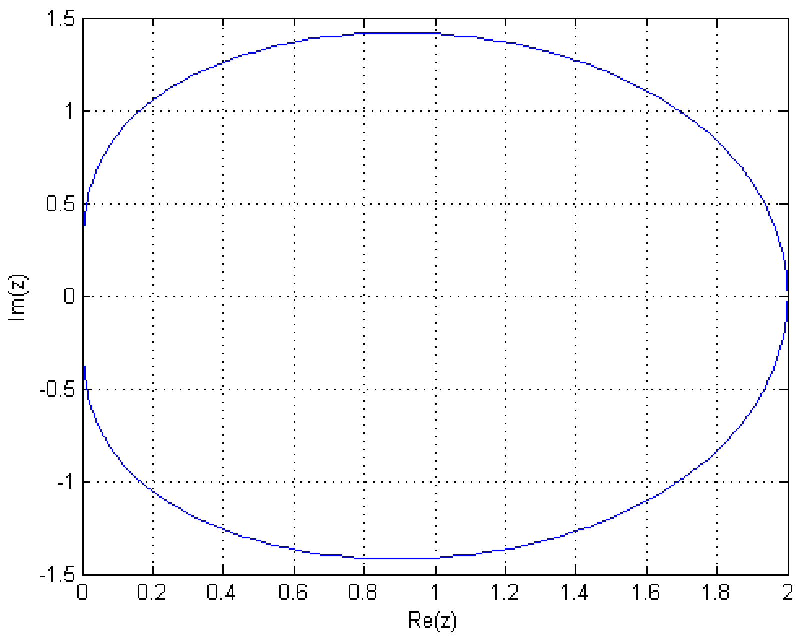

Figure 9: the region of absolute stability of the block Adams Moulton Backward Differential Formula for step number k=4

3. Numerical Results

In this section, some stiff initial value problems of Ordinary Differential Equations was solved using the discrete schemes derived in the previous chapter in block form to observe their performance. The results are obtained using Maple16.

Test Problem 1

$y_1' = 998y_1 + 1998y_2$
$y_2' = -999y_1 + 1999y_2$
$y_1(0) = 1, \qquad h = 0.1$
$y_2(0) = 0, \qquad x \in [0,3]$
Exact solution
$y_1(x) = 2e^{-x} - e^{-1000x}$
$y_2(x) = -e^{-x} - e^{-1000x}$

Table 1: Solution and Errors of Problem 1, Using the Adams Moulton Backward Differential formula k=2

x	Exact y1	Exact y2	Numerical y1	Numerical y2	Error y1	Error y2
0.1	1.81E+00	-9.05E-01	2.05E+00	-1.15E+00	2.45E-01	2.45E-01
0.2	1.64E+00	-8.19E-01	1.63E+00	-8.10E-01	9.18E-03	9.18E-03
0.3	1.48E+00	-7.41E-01	1.48E+00	-7.43E-01	2.25E-03	2.25E-03
0.4	1.34E+00	-6.70E-01	1.34E+00	-6.70E-01	8.46E-05	8.45E-05
0.5	1.21E+00	-6.07E-01	1.21E+00	-6.07E-01	1.95E-05	2.01E-05
0.6	1.10E+00	-5.49E-01	1.10E+00	-5.49E-01	1.16E-06	9.69E-07
0.7	9.93E-01	-4.97E-01	9.93E-01	-4.97E-01	8.64E-07	3.37E-07
0.8	8.99E-01	-4.49E-01	8.99E-01	-4.49E-01	4.37E-07	2.22E-07
0.9	8.13E-01	-4.07E-01	8.13E-01	-4.07E-01	9.60E-07	4.79E-07
1	7.36E-01	-3.68E-01	7.36E-01	-3.68E-01	4.41E-07	2.21E-07
1.1	6.66E-01	-3.33E-01	6.66E-01	-3.33E-01	8.68E-07	4.34E-07
1.2	6.02E-01	-3.01E-01	6.02E-01	-3.01E-01	4.35E-07	2.18E-07
1.3	5.45E-01	-2.73E-01	5.45E-01	-2.73E-01	7.77E-07	3.89E-07
1.4	4.93E-01	-2.47E-01	4.93E-01	-2.47E-01	4.16E-07	2.08E-07
1.5	4.46E-01	-2.23E-01	4.46E-01	-2.23E-01	6.91E-07	3.45E-07
1.6	4.04E-01	-2.02E-01	4.04E-01	-2.02E-01	3.89E-07	1.95E-07
1.7	3.65E-01	-1.83E-01	3.65E-01	-1.83E-01	6.10E-07	3.05E-07
1.8	3.31E-01	-1.65E-01	3.31E-01	-1.65E-01	3.59E-07	1.80E-07
1.9	2.99E-01	-1.50E-01	2.99E-01	-1.50E-01	5.36E-07	2.68E-07
2	2.71E-01	-1.35E-01	2.71E-01	-1.35E-01	3.27E-07	1.64E-07
2.1	2.45E-01	-1.22E-01	2.45E-01	-1.22E-01	4.69E-07	2.35E-07
2.2	2.22E-01	-1.11E-01	2.22E-01	-1.11E-01	2.95E-07	1.48E-07
2.3	2.01E-01	-1.00E-01	2.01E-01	-1.00E-01	4.08E-07	2.04E-07
2.4	1.81E-01	-9.07E-02	1.81E-01	-9.07E-02	2.64E-07	1.32E-07
2.5	1.64E-01	-8.21E-02	1.64E-01	-8.21E-02	3.54E-07	1.77E-07
2.6	1.49E-01	-7.43E-02	1.49E-01	-7.43E-02	2.34E-07	1.17E-07
2.7	1.34E-01	-6.72E-02	1.34E-01	-6.72E-02	3.06E-07	1.53E-07
2.8	1.22E-01	-6.08E-02	1.22E-01	-6.08E-02	2.06E-07	1.03E-07
2.9	1.10E-01	-5.50E-02	1.10E-01	-5.50E-02	2.64E-07	1.32E-07
3	9.96E-02	-4.98E-02	9.96E-02	-4.98E-02	1.81E-07	9.05E-08

Observed CPU time during implementation (2.25s)

Table 2: Solution and Errors of Problem 1, Using the Adams Moulton Backward Differential formula k=3

x	Exact y1	Exact y2	Numerical y1	Numerical y2	Error y1	Error y2
0.1	1.81E+00	-9.05E-01	9.29E-01	-5.32E-01	3.31E-03	3.34E-03
0.2	1.64E+00	-8.19E-01	9.53E-01	-9.53E-01	1.68E-03	1.66E-03
0.3	1.48E+00	-7.41E-01	9.29E-01	-7.78E-02	3.13E-03	3.15E-03
0.4	1.34E+00	-6.70E-01	8.64E-01	-4.68E-01	7.45E-05	3.19E-05
0.5	1.21E+00	-6.07E-01	8.85E-01	-9.85E-01	7.75E-05	4.13E-05
0.6	1.10E+00	-5.49E-01	8.64E-01	1.45E-02	6.01E-05	2.50E-05
0.7	9.93E-01	-4.97E-01	8.02E-01	-4.63E-01	9.48E-05	4.74E-05
0.8	8.99E-01	-4.49E-01	8.23E-01	-8.23E-01	8.23E-05	4.12E-05
0.9	8.13E-01	-4.07E-01	8.02E-01	-6.76E-02	7.80E-05	3.90E-05
1	7.36E-01	-3.68E-01	7.46E-01	-4.12E-01	9.39E-05	4.69E-05
1.1	6.66E-01	-3.33E-01	7.65E-01	-7.65E-01	8.23E-05	4.12E-05
1.2	6.02E-01	-3.01E-01	7.46E-01	-6.03E-02	7.71E-05	3.86E-05
1.3	5.45E-01	-2.73E-01	6.93E-01	-3.83E-01	8.70E-05	4.35E-05
1.4	4.93E-01	-2.47E-01	7.11E-01	-7.10E-01	7.68E-05	3.84E-05
1.5	4.46E-01	-2.23E-01	6.93E-01	-5.61E-02	7.15E-05	3.57E-05
1.6	4.04E-01	-2.02E-01	6.44E-01	-3.56E-01	7.74E-05	3.87E-05
1.7	3.65E-01	-1.83E-01	6.60E-01	-6.60E-01	6.86E-05	3.43E-05
1.8	3.31E-01	-1.65E-01	6.44E-01	-5.21E-02	6.35E-05	3.18E-05
1.9	2.99E-01	-1.50E-01	5.98E-01	-3.30E-01	6.69E-05	3.35E-05
2	2.71E-01	-1.35E-01	6.14E-01	-6.23E-01	5.95E-05	2.97E-05
2.1	2.45E-01	-1.22E-01	5.98E-01	-4.00E-02	5.49E-05	2.75E-05
2.2	2.22E-01	-1.11E-01	5.56E-01	-3.09E-01	5.67E-05	2.83E-05
2.3	2.01E-01	-1.00E-01	5.70E-01	-5.80E-01	5.05E-05	1.06E-02
2.4	1.81E-01	-9.07E-02	5.56E-01	-3.68E-02	4.65E-05	2.32E-05
2.5	1.64E-01	-8.21E-02	5.17E-01	-2.88E-01	4.73E-05	2.36E-05
2.6	1.49E-01	-7.43E-02	5.30E-01	-5.30E-01	4.22E-05	2.11E-05
2.7	1.34E-01	-6.72E-02	5.17E-01	-4.21E-02	3.88E-05	1.94E-05
2.8	1.22E-01	-6.08E-02	4.80E-01	-2.66E-01	3.89E-05	1.95E-05
2.9	1.10E-01	-5.50E-02	4.92E-01	-4.92E-01	3.48E-05	1.74E-05
3	9.96E-02	-4.98E-02	4.80E-01	-3.89E-02	3.19E-05	1.60E-05

Observed CPU time during implementation (2.45s)

Table 3: Solution and Errors of Problem 1, Using the Adams Moulton Backward Differential formula k=4

x	Exact y1	Exact y2	Numerical y1	Numerical y2	Error y1	Error y2
0.1	1.81E+00	-9.05E-01	5.23E-01	-3.23E-01	4.42E-03	3.32E-03
0.2	1.64E+00	-8.19E-01	4.32E-01	-9.79E-01	2.76E-03	1.65E-03
0.3	1.48E+00	-7.41E-01	9.29E-01	-7.24E-02	4.13E-03	3.24E-03
0.4	1.34E+00	-6.70E-01	8.64E-01	-4.68E-01	8.40E-04	3.63E-04
0.5	1.21E+00	-6.07E-01	7.15E-01	-9.85E-01	8.71E-04	4.12E-04
0.6	1.10E+00	-5.49E-01	8.64E-01	-1.45E-02	7.05E-04	2.54E-04
0.7	9.93E-01	-4.97E-01	6.02E-01	-3.23E-01	9.38E-05	4.64E-05
0.8	8.99E-01	-4.49E-01	8.23E-01	-8.01E-01	7.42E-05	4.22E-05
0.9	8.13E-01	-4.07E-01	8.02E-01	-5.61E-02	6.90E-05	3.41E-05
1	7.36E-01	-3.68E-01	7.42E-01	-6.14E-01	9.31E-05	4.65E-05
1.1	6.66E-01	-3.33E-01	6.05E-01	-6.44E-01	7.32E-05	4.19E-05
1.2	6.02E-01	-3.01E-01	7.46E-01	-7.41E-02	6.82E-05	3.56E-05
1.3	5.45E-01	-2.73E-01	6.14E-01	-3.92E-01	9.90E-05	4.75E-05
1.4	4.93E-01	-2.47E-01	7.55E-01	-6.12E-01	6.68E-05	3.94E-05
1.5	4.46E-01	-2.23E-01	6.73E-01	-4.52E-02	6.15E-05	3.62E-05
1.6	4.04E-01	-2.02E-01	6.74E-01	-3.65E-01	6.74E-05	3.88E-05
1.7	3.65E-01	-1.83E-01	5.30E-01	-7.70E-01	5.86E-05	3.42E-05
1.8	3.31E-01	-1.65E-01	5.81E-01	-5.31E-02	5.22E-05	3.15E-05
1.9	2.99E-01	-1.50E-01	5.89E-01	-3.42E-01	5.74E-05	3.41E-05
2	2.71E-01	-1.35E-01	5.74E-01	-4.47E-01	4.74E-05	2.27E-05
2.1	2.45E-01	-1.22E-01	4.98E-01	-4.31E-02	4.38E-05	2.45E-05
2.2	2.22E-01	-1.11E-01	4.65E-01	-3.09E-01	4.54E-05	2.51E-05
2.3	2.01E-01	-1.00E-01	4.90E-01	-5.80E-01	4.01E-05	1.07E-02
2.4	1.81E-01	-9.07E-02	4.26E-01	-3.68E-02	3.48E-05	2.21E-05
2.5	1.64E-01	-8.21E-02	4.24E-01	-2.88E-01	3.42E-05	2.66E-05
2.6	1.49E-01	-7.43E-02	4.41E-01	-4.40E-01	3.11E-05	2.24E-05
2.7	1.34E-01	-6.72E-02	4.14E-01	-5.31E-02	2.74E-05	1.94E-05
2.8	1.22E-01	-6.08E-02	3.80E-01	-1.47E-01	2.62E-05	1.95E-05
2.9	1.10E-01	-5.50E-02	3.72E-01	-3.71E-01	4.22E-05	1.74E-05
3	9.96E-02	-4.98E-02	3.70E-01	-3.71E-02	4.12E-05	1.65E-05

Observed CPU time during implementation (2.64s)

Test Problem 2

$y_1' = -0.1y_1 - 199.9y_2$
$y_2' = -200y_2$
$y_1(0) = 2, \quad h = 0.1$
$y_2(0) = 1, \quad x \in [0,3]$
Exact solution
$y_1(x) = e^{-200x} + e^{-0.1x}$
$y_2(x) = e^{-200x}$

Table 4: Solution and Error of Problem 2, Using the Adams Moulton Backward Differential Formula k=2

x	Exact y1	Exact y2	Numerical y1	Numerical y2	Error y1	Error y2
0.1	9.90E-01	2.06E-09	7.68E-01	-2.22E-01	2.22E-01	2.22E-01
0.2	9.80E-01	4.25E-18	1.01E+00	3.24E-02	3.24E-02	3.24E-02
0.3	9.70E-01	8.76E-27	9.63E-01	-7.18E-05	7.18E-03	7.18E-05
0.4	9.61E-01	1.80E-35	9.62E-01	1.05E-05	1.05E-03	1.05E-05
0.5	9.51E-01	3.72E-44	9.51E-01	-2.33E-07	2.33E-04	2.33E-07
0.6	9.42E-01	7.67E-53	9.42E-01	3.40E-10	3.40E-05	3.40E-10
0.7	9.32E-01	1.58E-61	9.32E-01	-7.54E-13	7.54E-06	7.54E-13
0.8	9.23E-01	3.26E-70	9.23E-01	1.10E-16	1.10E-06	1.10E-16
0.9	9.14E-01	6.71E-79	9.14E-01	-2.44E-18	2.48E-07	2.44E-18
1	9.05E-01	1.38E-87	9.05E-01	3.57E-18	3.14E-08	3.57E-18
1.1	8.96E-01	2.85E-96	8.96E-01	-7.91E-19	1.25E-08	7.91E-19
1.2	8.87E-01	5.88E-105	8.87E-01	1.15E-22	3.92E-09	1.15E-22
1.3	8.78E-01	1.21E-113	8.78E-01	-2.56E-23	5.62E-09	2.56E-23
1.4	8.69E-01	2.50E-122	8.69E-01	3.74E-23	5.70E-09	3.74E-23
1.5	8.61E-01	5.15E-131	8.61E-01	-8.30E-25	5.93E-09	8.30E-25
1.6	8.52E-01	1.06E-139	8.52E-01	1.21E-27	6.27E-09	1.21E-27
1.7	8.44E-01	2.19E-148	8.44E-01	-2.69E-28	6.60E-09	2.69E-28
1.8	8.35E-01	4.51E-157	8.35E-01	3.92E-29	6.91E-09	3.92E-29
1.9	8.27E-01	9.29E-166	8.27E-01	-8.70E-30	7.24E-09	8.70E-30
2	8.19E-01	1.92E-174	8.19E-01	1.27E-31	7.58E-09	1.27E-31
2.1	8.11E-01	3.95E-183	8.11E-01	-2.82E-32	7.77E-09	2.82E-32
2.2	8.03E-01	8.14E-192	8.03E-01	4.12E-35	8.06E-09	4.12E-35
2.3	7.95E-01	1.68E-200	7.95E-01	-9.13E-36	8.30E-09	9.13E-36
2.4	7.87E-01	3.46E-209	7.87E-01	1.33E-37	8.57E-09	1.33E-37
2.5	7.79E-01	7.12E-218	7.79E-01	-2.96E-39	8.87E-09	2.96E-39
2.6	7.71E-01	1.47E-226	7.71E-01	4.32E-43	9.10E-09	4.32E-43
2.7	7.63E-01	3.03E-235	7.63E-01	-9.58E-45	9.34E-09	9.58E-45
2.8	7.56E-01	6.24E-244	7.56E-01	1.40E-48	9.66E-09	1.40E-48
2.9	7.48E-01	1.29E-252	7.48E-01	-3.10E-50	9.88E-09	3.10E-50
3	7.41E-01	2.65E-261	7.41E-01	4.53E-52	9.95E-09	4.53E-52

Observed CPU time during implementation (1.93s)

Table 5: Solution and Error of Problem 2, Using the Adams Moulton Backward Differential formula k=3

x	Exact y1	Exact y2	Numerical y1	Numerical y2	Error y1	Error y2
0.1	9.90E-01	2.06E-09	7.68E-01	-2.22E-01	1.94E-01	1.94E-01
0.2	9.80E-01	4.25E-18	1.01E+00	3.24E-02	9.18E-02	9.18E-02
0.3	9.70E-01	8.76E-27	9.63E-01	-7.18E-05	2.02E-02	2.02E-02
0.4	9.61E-01	1.80E-35	9.62E-01	1.05E-05	3.93E-03	3.93E-03
0.5	9.51E-01	3.72E-44	9.51E-01	-2.33E-07	1.86E-03	1.86E-03
0.6	9.42E-01	7.67E-53	9.42E-01	3.40E-10	4.09E-04	4.09E-04
0.7	9.32E-01	1.58E-61	9.32E-01	-7.54E-13	7.94E-05	7.94E-05
0.8	9.23E-01	3.26E-70	9.23E-01	1.10E-16	3.75E-05	3.75E-05
0.9	9.14E-01	6.71E-79	9.14E-01	-2.44E-18	8.27E-06	8.27E-06
1	9.05E-01	1.38E-87	9.05E-01	3.57E-18	1.60E-06	1.61E-06
1.1	8.96E-01	2.85E-96	8.96E-01	-7.91E-19	7.61E-07	7.59E-07
1.2	8.87E-01	5.88E-105	8.87E-01	1.15E-22	1.65E-07	1.67E-07
1.3	8.78E-01	1.21E-113	8.78E-01	-2.56E-23	3.45E-08	3.25E-08
1.4	8.69E-01	2.50E-122	8.69E-01	3.74E-23	1.33E-08	1.54E-08
1.5	8.61E-01	5.15E-131	8.61E-01	-8.30E-25	5.73E-09	3.38E-09
1.6	8.52E-01	1.06E-139	8.52E-01	1.21E-27	1.57E-09	6.57E-10
1.7	8.44E-01	2.19E-148	8.44E-01	-2.69E-28	2.60E-09	3.11E-10
1.8	8.35E-01	4.51E-157	8.35E-01	3.92E-29	2.51E-09	6.84E-11
1.9	8.27E-01	9.29E-166	8.27E-01	-8.70E-30	2.24E-09	1.33E-11
2	8.19E-01	1.92E-174	8.19E-01	1.27E-31	2.28E-09	6.28E-12
2.1	8.11E-01	3.95E-183	8.11E-01	-2.82E-32	2.57E-09	1.38E-12
2.2	8.03E-01	8.14E-192	8.03E-01	4.12E-35	2.16E-09	2.69E-13
2.3	7.95E-01	1.68E-200	7.95E-01	-9.13E-36	2.20E-09	1.27E-14
2.4	7.87E-01	3.46E-209	7.87E-01	1.33E-37	2.37E-09	7.98E-15
2.5	7.79E-01	7.12E-218	7.79E-01	-2.96E-39	2.17E-09	5.43E-16
2.6	7.71E-01	1.47E-226	7.71E-01	4.32E-43	2.10E-09	2.57E-17
2.7	7.63E-01	3.03E-235	7.63E-01	-9.58E-45	2.34E-09	5.66E-18
2.8	7.56E-01	6.24E-244	7.56E-01	1.40E-48	2.16E-09	1.10E-19
2.9	7.48E-01	1.29E-252	7.48E-01	-3.10E-50	2.18E-09	5.19E-20
3	7.41E-01	2.65E-261	7.41E-01	4.53E-52	2.38E-09	1.14E-21

Observed CPU time during implementation (1.93s)

Table 6: Solution and Error of Problem 2, Using the Adams Moulton Backward Differential formula k=4

X	Exact y1	Exact y2	Numerical y1	Numerical y2	Error y1	Error y2
0.1	9.90E-01	2.06E-09	7.68E-01	-2.22E-01	1.94E-01	1.87E-01
0.2	9.80E-01	4.25E-18	1.01E+00	3.24E-02	9.18E-02	8.12E-02
0.3	9.70E-01	8.76E-27	9.63E-01	-7.18E-05	2.02E-02	3.02E-02
0.4	9.61E-01	1.80E-35	9.62E-01	1.05E-05	2.93E-03	2.99E-03
0.5	9.51E-01	3.72E-44	9.51E-01	-2.33E-07	1.86E-03	1.78E-03
0.6	9.42E-01	7.67E-53	9.42E-01	3.40E-10	3.09E-04	5.07E-04
0.7	9.32E-01	1.58E-61	9.32E-01	-7.54E-13	7.94E-05	6.90E-05
0.8	9.23E-01	3.26E-70	9.23E-01	1.10E-16	3.75E-05	4.57E-05
0.9	9.14E-01	6.71E-79	9.14E-01	-2.44E-18	8.27E-06	7.34E-06
1	9.05E-01	1.38E-87	9.05E-01	3.57E-18	1.60E-06	2.51E-06
1.1	8.96E-01	2.85E-96	8.96E-01	-7.91E-19	7.61E-07	7.62E-07
1.2	8.87E-01	5.88E-105	8.87E-01	1.15E-22	1.65E-07	1.59E-07
1.3	8.78E-01	1.21E-113	8.78E-01	-2.56E-23	3.45E-08	2.20E-08
1.4	8.69E-01	2.50E-122	8.69E-01	3.74E-23	1.33E-08	3.54E-08
1.5	8.61E-01	5.15E-131	8.61E-01	-8.30E-25	5.73E-09	1.12E-09
1.6	8.52E-01	1.06E-139	8.52E-01	1.21E-27	1.57E-09	6.21E-09
1.7	8.44E-01	2.19E-148	8.44E-01	-2.69E-28	3.60E-09	3.10E-09
1.8	8.35E-01	4.51E-157	8.35E-01	3.92E-29	3.51E-09	6.87E-10
1.9	8.27E-01	9.29E-166	8.27E-01	-8.70E-30	3.24E-09	1.03E-10
2	8.19E-01	1.92E-174	8.19E-01	1.27E-31	3.28E-09	6.42E-11
2.1	8.11E-01	3.95E-183	8.11E-01	-2.82E-32	3.57E-09	1.38E-11
2.2	8.03E-01	8.14E-192	8.03E-01	4.12E-35	2.16E-09	2.69E-12
2.3	7.95E-01	1.68E-200	7.95E-01	-9.13E-36	2.14E-09	1.27E-13
2.4	7.87E-01	3.46E-209	7.87E-01	1.33E-37	2.21E-09	6.88E-14
2.5	7.79E-01	7.12E-218	7.79E-01	-2.96E-39	2.14E-09	4.32E-15
2.6	7.71E-01	1.47E-226	7.71E-01	4.32E-43	2.20E-09	2.65E-16
2.7	7.63E-01	3.03E-235	7.63E-01	-9.58E-45	2.30E-09	5.77E-17
2.8	7.56E-01	6.24E-244	7.56E-01	1.40E-48	2.19E-09	2.12E-18
2.9	7.48E-01	1.29E-252	7.48E-01	-3.10E-50	2.21E-09	4.09E-19
3	7.41E-01	2.65E-261	7.41E-01	4.53E-52	2.39E-09	1.23E-19

Observed CPU time during implementation (1.95s)

Test Problem 3

$y_1' = -0.1y_1 - 49.9y_2$
$y_2' = -50y_2$
$y_3' = 70y_2 - 120y_3$
$y_1(0) = 2,$
$y_2(0) = 1, \qquad h = 0.01$
$y_3(0) = 2, \qquad x \in [0,3]$
Exact solution
$y_1(x) = e^{-0.1x} + e^{-50x}$
$y_2(x) = e^{-200x}$
$y_3(x) = e^{-50x} + e^{-120x}$

Table 7: Solution and Error of Problem 3, Using the Adams Moulton Backward Differential Formula k=2

x	Exact y1	Exact y2	Exact y3	Numerical y1	Numerical y2	Numerical y3	Error y1	Error y2	Error y3
0.1	9.97E-01	2.06E-09	6.74E-03	2.15E-01	5.27E-01	1.26E+00	7.82E-01	5.27E-01	1.25E+00
0.2	9.80E-01	4.25E-18	4.54E-05	5.66E-02	-1.93E-01	-4.79E-01	9.24E-01	1.93E-01	4.79E-01
0.3	9.70E-01	8.76E-27	3.06E-07	5.62E-03	8.37E-02	2.01E-01	9.65E-01	8.37E-02	2.01E-01
0.4	9.61E-01	1.80E-35	2.06E-09	2.34E-03	-6.01E-02	-1.42E-01	9.58E-01	6.01E-02	1.42E-01
0.5	9.51E-01	3.72E-44	1.39E-11	1.24E-04	1.96E-02	4.60E-02	9.51E-01	1.96E-02	4.60E-02
0.6	9.42E-01	7.67E-53	9.36E-14	2.69E-04	-1.53E-02	-3.50E-02	9.41E-01	1.53E-02	3.50E-02
0.7	9.32E-01	1.58E-61	6.31E-16	-3.13E-06	4.88E-03	1.11E-02	9.32E-01	4.88E-03	1.11E-02
0.8	9.23E-01	3.26E-70	4.25E-18	5.88E-05	-3.84E-03	-8.49E-03	9.23E-01	3.84E-03	8.49E-03
0.9	9.14E-01	6.71E-79	2.86E-20	-1.70E-06	1.22E-03	2.67E-03	9.14E-01	1.22E-03	2.67E-03
1	9.05E-01	1.38E-87	1.93E-22	1.45E-05	-9.61E-04	-2.05E-04	9.05E-01	9.61E-04	2.05E-04
1.1	8.96E-01	2.85E-96	1.30E-24	-4.48E-07	3.04E-04	6.45E-04	8.96E-01	3.04E-04	6.45E-04
1.2	8.87E-01	5.88E-105	8.76E-27	3.61E-06	-2.40E-05	-4.95E-05	8.87E-01	2.40E-05	4.95E-05
1.3	8.78E-01	1.21E-113	5.90E-29	-1.13E-07	7.61E-06	1.56E-06	8.78E-01	7.61E-06	1.56E-06
1.4	8.69E-01	2.50E-122	3.98E-31	9.02E-07	-6.00E-07	-1.19E-07	8.69E-01	6.00E-07	1.19E-07
1.5	8.61E-01	5.15E-131	2.68E-33	-2.82E-08	1.90E-09	3.75E-09	8.61E-01	1.90E-09	3.75E-09
1.6	8.52E-01	1.06E-139	1.80E-35	2.25E-07	-1.50E-09	-2.87E-09	8.52E-01	1.50E-09	2.87E-09
1.7	8.44E-01	2.19E-148	1.22E-37	-7.05E-09	4.75E-10	9.04E-10	8.44E-01	4.75E-10	9.04E-10
1.8	8.35E-01	4.51E-157	8.19E-40	5.64E-08	-3.75E-11	-6.89E-11	8.35E-01	3.75E-11	6.89E-11
1.9	8.27E-01	9.29E-166	5.52E-42	-1.76E-09	1.19E-12	2.17E-12	8.27E-01	1.19E-12	2.17E-12
2	8.19E-01	1.92E-174	3.72E-44	1.41E-08	-9.37E-14	-1.65E-14	8.19E-01	9.37E-14	1.65E-14
2.1	8.11E-01	3.95E-183	2.51E-46	-4.40E-10	2.97E-16	5.21E-16	8.11E-01	2.97E-16	5.21E-16
2.2	8.03E-01	8.14E-192	1.69E-48	3.52E-09	-2.34E-18	-3.96E-18	8.03E-01	2.34E-18	3.96E-18
2.3	7.95E-01	1.68E-200	1.14E-50	-1.10E-10	7.42E-19	1.25E-19	7.95E-01	7.42E-19	1.25E-19
2.4	7.87E-01	3.46E-209	7.67E-53	8.80E-10	-5.85E-20	-9.46E-20	7.87E-01	5.85E-20	9.46E-20
2.5	7.79E-01	7.12E-218	5.17E-55	-2.75E-11	1.85E-21	2.98E-21	7.79E-02	1.85E-21	2.98E-21
2.6	7.71E-01	1.47E-226	3.48E-57	2.20E-10	-1.46E-22	-2.25E-22	7.71E-02	1.46E-22	2.25E-22
2.7	7.63E-01	3.03E-235	2.35E-59	-6.86E-12	4.63E-23	7.12E-23	7.63E-02	4.63E-23	7.12E-23
2.8	7.56E-01	6.24E-244	1.58E-61	5.50E-11	-3.66E-24	-5.36E-24	7.59E-02	3.66E-24	5.36E-24
2.9	7.48E-01	1.29E-252	1.06E-63	-1.72E-12	1.16E-25	1.69E-25	7.78E-02	1.16E-25	1.69E-25
3	7.41E-01	2.65E-261	7.18E-66	1.37E-11	-9.14E-26	-1.27E-26	7.74E-02	9.14E-26	1.27E-26

Observed CPU time during implementation (2.40s)

Table 8: Solution and Error of Problem 3, Using the Adams Moulton Backward Differential Formula k=3

X	Exact y1	Exact y2	Exact y3	Numerical y1	Numerical y2	Numerical y3	Error y1	Error y2	Error y3
0.1	9.97E-01	2.06E-09	6.74E-03	1.61E+00	6.06E-01	3.70E-01	6.09E-01	6.06E-01	3.63E-01
0.2	9.80E-01	4.25E-18	4.54E-05	1.37E+00	3.68E-01	5.10E-01	3.86E-01	3.68E-01	5.10E-01
0.3	9.70E-01	8.76E-27	3.06E-07	1.22E+00	2.23E-01	8.60E-02	2.50E-01	2.23E-01	8.60E-02
0.4	9.61E-01	1.80E-35	2.06E-09	1.13E+00	1.35E-01	1.69E-01	1.71E-01	1.35E-01	1.69E-01
0.5	9.51E-01	3.72E-44	1.39E-11	1.08E+00	8.21E-02	6.34E-02	1.26E-01	8.21E-02	6.34E-02
0.6	9.42E-01	7.67E-53	9.36E-14	1.04E+00	4.98E-02	6.95E-02	1.02E-01	4.98E-02	6.95E-02
0.7	9.32E-01	1.58E-61	6.31E-16	1.02E+00	3.02E-02	2.65E-02	9.08E-02	3.02E-02	2.65E-02
0.8	9.23E-01	3.26E-70	4.25E-18	1.01E+00	1.83E-02	2.18E-02	8.72E-02	1.83E-02	2.18E-02
0.9	9.14E-01	6.71E-79	2.86E-20	1.00E+00	1.11E-02	9.23E-03	8.82E-02	1.11E-02	9.23E-03
1	9.05E-01	1.38E-87	1.93E-22	9.97E-01	6.74E-03	8.19E-03	9.19E-02	6.74E-03	8.19E-03
1.1	8.96E-01	2.85E-96	1.30E-24	9.93E-01	4.09E-03	4.55E-03	9.73E-02	4.09E-03	4.55E-03
1.2	8.87E-01	5.88E-105	8.76E-27	9.91E-01	2.48E-03	3.59E-03	1.04E-01	2.48E-03	3.59E-03
1.3	8.78E-01	1.21E-113	5.90E-29	9.89E-01	1.50E-03	2.25E-03	1.10E-01	1.50E-03	2.25E-03
1.4	8.69E-01	2.50E-122	3.98E-31	9.87E-01	9.11E-04	1.79E-04	1.18E-01	9.11E-04	1.79E-04
1.5	8.61E-01	5.15E-131	2.68E-33	9.86E-01	5.53E-04	1.25E-04	1.25E-01	5.53E-04	1.25E-04
1.6	8.52E-01	1.06E-139	1.80E-35	9.84E-01	3.35E-05	1.18E-04	1.32E-01	3.35E-05	1.18E-04
1.7	8.44E-01	2.19E-148	1.22E-37	9.83E-01	2.03E-05	1.02E-05	1.40E-01	2.03E-05	1.02E-05
1.8	8.35E-01	4.51E-157	8.19E-40	9.82E-01	1.23E-06	8.74E-05	1.47E-01	1.23E-06	8.74E-05
1.9	8.27E-01	9.29E-166	5.52E-42	9.81E-01	7.48E-06	8.99E-05	1.54E-01	7.48E-06	8.99E-05
2	8.19E-01	1.92E-174	3.72E-44	9.80E-01	4.54E-07	8.71E-06	1.62E-01	4.54E-07	8.71E-06
2.1	8.11E-01	3.95E-183	2.51E-46	9.79E-01	2.75E-07	7.69E-06	1.69E-01	2.75E-07	7.69E-06
2.2	8.03E-01	8.14E-192	1.69E-48	9.78E-01	1.67E-08	8.40E-06	1.76E-01	1.67E-08	8.40E-06
2.3	7.95E-01	1.68E-200	1.14E-50	9.77E-01	1.01E-08	8.33E-07	1.83E-01	1.01E-08	8.33E-07
2.4	7.87E-01	3.46E-209	7.67E-53	9.76E-01	6.14E-09	7.46E-07	1.90E-01	6.14E-09	7.46E-07
2.5	7.79E-01	7.12E-218	5.17E-55	9.75E-01	3.72E-09	8.25E-07	1.97E-01	3.72E-09	8.25E-07
2.6	7.71E-01	1.47E-226	3.48E-57	9.74E-02	2.26E-10	8.22E-08	6.74E-02	2.26E-10	8.22E-08
2.7	7.63E-01	3.03E-235	2.35E-59	9.73E-02	1.37E-10	7.39E-08	6.66E-02	1.37E-10	7.39E-08
2.8	7.56E-01	6.24E-244	1.58E-61	9.72E-02	8.31E-11	8.19E-08	6.62E-02	8.31E-11	8.19E-08
2.9	7.48E-01	1.29E-252	1.06E-63	9.71E-02	5.04E-11	8.18E-09	6.51E-02	5.04E-11	8.18E-09
3	7.41E-01	2.65E-261	7.18E-66	9.70E-02	3.06E-12	7.36E-09	6.44E-02	3.06E-12	7.36E-09

Observed CPU time during implementation (2.68s)

Table 9: Solution and Error of Problem 3, Using the Adams Moulton Backward Differential Formula k=4

X	Exact y1	Exact y2	Exact y3	Numerical y1	Numerical y2	Numerical y3	Error y1	Error y2	Error y3
0.1	9.97E-01	2.06E-09	6.74E-03	3.15E+00	7.16E-01	5.41E-01	6.09E-01	6.06E-01	5.24E-01
0.2	9.80E-01	4.25E-18	4.54E-05	3.37E+00	5.68E-01	6.09E-01	3.86E-01	3.68E-01	7.16E-01
0.3	9.70E-01	8.76E-27	3.06E-07	5.22E+00	2.31E-01	4.40E-01	2.50E-01	2.23E-01	9.71E-02
0.4	9.61E-01	1.80E-35	2.06E-09	2.13E+00	1.22E-01	3.63E-02	1.71E-01	1.35E-01	2.69E-01
0.5	9.51E-01	3.72E-44	1.39E-11	2.08E+00	7.42E-02	4.31E-01	1.26E-01	8.21E-02	5.41E-02
0.6	9.42E-01	7.67E-53	9.36E-14	1.04E+00	5.98E-02	5.72E-02	1.02E-01	4.98E-02	7.90E-02
0.7	9.32E-01	1.58E-61	6.31E-16	1.02E+00	4.04E-02	4.61E-02	9.08E-02	3.02E-02	2.65E-02
0.8	9.23E-01	3.26E-70	4.25E-18	6.01E+00	3.72E-02	3.18E-02	8.72E-02	1.83E-02	3.18E-02
0.9	9.14E-01	6.71E-79	2.86E-20	4.00E+00	2.14E-02	9.32E-02	8.82E-02	1.11E-02	9.23E-03
1	9.05E-01	1.38E-87	1.93E-22	2.70E-01	1.42E-03	8.44E-02	9.19E-02	6.74E-03	7.19E-03
1.1	8.96E-01	2.85E-96	1.30E-24	2.71E-01	6.09E-03	5.44E-02	9.73E-02	4.09E-03	2.55E-03
1.2	8.87E-01	5.88E-105	8.76E-27	3.72E-01	2.48E-03	3.19E-03	1.04E-01	2.48E-04	3.59E-03
1.3	8.78E-01	1.21E-113	5.90E-29	7.73E-01	1.21E-03	2.25E-03	1.10E-01	1.50E-04	2.25E-03
1.4	8.69E-01	2.50E-122	3.98E-31	7.74E-01	9.03E-04	1.79E-03	1.18E-01	9.11E-05	1.79E-04
1.5	8.61E-01	5.15E-131	2.68E-33	7.75E-01	5.14E-04	1.25E-03	1.25E-01	5.53E-05	1.40E-04
1.6	8.52E-01	1.06E-139	1.80E-35	7.76E-02	3.43E-05	1.11E-03	1.32E-01	3.35E-06	1.18E-04
1.7	8.44E-01	2.19E-148	1.22E-37	8.77E-02	2.71E-05	1.08E-03	1.40E-01	2.03E-06	1.02E-05
1.8	8.35E-01	4.51E-157	8.19E-40	8.78E-01	1.23E-06	7.42E-03	1.47E-01	1.23E-07	9.74E-05
1.9	8.27E-01	9.29E-166	5.52E-42	8.79E-02	7.E-06	7.99E-03	1.54E-01	7.48E-07	9.99E-05
2	8.19E-01	1.92E-174	3.72E-44	8.80E-02	6.24E-07	7.71E-04	1.62E-01	5.62E-08	9.70E-06
2.1	8.11E-01	3.95E-183	2.51E-46	8.81E-02	2.75E-07	8.69E-04	1.69E-01	3.70E-08	8.96E-06
2.2	8.03E-01	8.14E-192	1.69E-48	9.82E-02	1.67E-08	7.40E-04	1.76E-01	1.66E-09	7.30E-07
2.3	7.95E-01	1.68E-200	1.14E-50	9.83E-02	1.01E-08	7.33E-05	1.83E-01	1.07E-09	7.29E-08
2.4	7.87E-01	3.46E-209	7.67E-53	9.84E-02	6.14E-09	6.46E-05	1.90E-01	6.17E-10	6.49E-09
2.5	7.79E-01	7.12E-218	5.17E-55	9.85E-02	3.72E-09	7.25E-05	1.97E-01	4.42E-10	9.41E-09
2.6	7.71E-01	1.47E-226	3.48E-57	9.86E-02	2.26E-10	7.22E-06	6.74E-02	2.26E-11	9.41E-10
2.7	7.63E-01	3.03E-235	2.35E-59	9.87E-03	1.37E-10	6.41E-06	6.66E-02	1.37E-11	7.20E-10
2.8	7.56E-01	6.24E-244	1.58E-61	6.88E-03	8.31E-11	8.12E-06	6.62E-02	7.32E-13	8.74E-11
2.9	7.48E-01	1.29E-252	1.06E-63	6.89E-03	5.04E-11	8.16E-07	6.51E-02	3.07E-13	9.17E-11
3	7.41E-01	2.65E-261	7.18E-66	6.90E-03	3.06E-12	7.21E-07	6.44E-02	4.41E-14	9.17E-12

Observed CPU time during implementation (2.80s)

Discussion

The numerical experiments carried out in this study provide important insights into the behaviour, efficiency and stability characteristics of the blended multistep methods developed from the Adams Moulton Backward Differentiation Framework. From the stability plots generated for the various schemes, it is evident that the regions of absolute stability vary with the order of the blended formulation. In particular, Adams Moulton Backward Differential Formula for k=3, 4 shows that they are A-stable, this implies that these schemes can reliably handle stiff initial value problems while Adams Moulton Backward Differential Formula k=2 are A(α) stable. Indicating a slightly restricted but still robust stability suitable for moderately stiff systems.

The numerical results obtained from Test Problem 1 clearly demonstrate that the blended method consistently outperformed the second derivative method especially in terms of error reduction and overall solution accuracy. This superior performance highlights the ability of the blended construction to capture local solution dynamics more efficiently by combining information from multiple approximation structures. However, this enhanced accuracy was accompanied by a noticeable increase in computational demand as reflected by longer CPU execution time.

For Test Problem 2, involving the analysis for k=2, the second derivative method initially performed competitively, but the blended form eventually produced more accurate approximations as the step size decreased. When the order was increased to k=3 and beyond, the blended formulations again demonstrated consistently lower errors compared to the corresponding second derivative scheme. The observed accuracy gains, however, were accompanied by a further increase in CPU time. This computational cost grows with method complexity, confirming that higher order blended methods demand more operations per step. Despite this, the improvement in accuracy may justify the additional cost, especially in applications where solution precision is more critical than computational speed.

For Test Problem 3, analysed for k=2, 3, and 4 further reinforced the findings above, in all cases, the blended methods produced highly accurate numerical solutions, outperforming the second derivative approach by a significant margin. These results indicate that the blended framework offers a more effective approximation.

An important observation from the comparative analysis is that the two-step blended scheme displayed better accuracy and stability characteristics than the three step and four step. Although higher step blended methods theoretically provide more information per iteration. The two-step blended method struck the most favourable balance between accuracy and computational efficiency, making it particularly well suited for stiff problems requiring both reliability and moderate computational cost.

Overall, the results across all test problems reveal a consistent pattern, blended methods form a powerful numerical approximation framework capable of delivering high accuracy for stiff and non-stiff initial value problems. Their stability properties make them suitable for long time integrations, while their improved accuracy ensures reliable solution behaviour even under small step sizes. However, the computational overhead associated with blended formulations must be considered when selecting an appropriate numerical scheme. In applications where processor time is limited or where real time computations are required, the second derivative method may remain a viable alternative. Conversely in high accuracy settings such as engineering simulations, scientific modelling and stiff system analysis, the blended method developed in this paper offer a more desirable and robust solution approach despite the higher CPU cost

5. Conclusion

The construction of the discrete schemes of the method were deduced from their respective continuous formulations. All discrete schemes were found to be convergent. The analysis of the stability properties shows that some methods are A-stable while others are A(α)-stable. It was observed in tables 1-9, that the blended forms performed better than the second derivative for the problems solved, but at a higher cost of CPU time.

In particular, the two-step blended form are more accurate than the three and four step blended forms, making it a preferable option for solving stiff initial value problems where high accuracy is essential.

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