SHADOW PRICES FOR INFINITE GROWTH PROGRAMS: THE FUNCTIONAL ANALYSIS APPROACH

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ABSTRACT

This paper summarizes some recent results on price systems characterizing efficient or optimal programs in economic models for which the commodity space can be represented as a subset of L_∞. In general, a price system is represented as a continuous linear function on L_∞. An efficient program is called regular if there is an associated price system of the form $P(y) = \int gy$, where g is in L_1. Various conditions are given under which efficient programs can be approximated by regular programs. The approach is illustrated by examples of economic optimization problems involving time and uncertainty.

Keywords: *optimal programs, economic models, commodity space, economic optimization problems.*

INTRODUCTION

Some of the most intensively studied economic problems center around characterization of efficient programs of resource allocation by a price system, and the use of a price mechanism to attain such an allocation in economics in which decision- making is decentralized. A well- known proposition is that, under an appropriate convexity assumption on technology, an efficient production program maximizes the value of net output if value is calculated using a proper system of prices. The extension of the standard theory to programs involving infinite numbers of "commodities" poses certain mathematical difficulties, which in turn raises deeper conception problems concerning the proper definition of a "price system" when the commodity space is infinite dimensional. Alternative approaches have been suggested by different writers (see Debreu, 2004; Hurwicz, 2008; Malinvand, 2003; and Peleg and Yaari, 2000); and in the present paper we survey some recent results obtained with the tools of functional analysis.

From the point of view of Mathematical Programming or Control Theory, prices can be identified with "Langrangean Multipliers" or "Dual Variables". In addition to suggesting how economic decisions can be decentralized, the study of price systems associated with efficient or optimal programs has been used to provide qualitative information about such programs, e.g., about existence and asymptotic properties (see, e.g., the collection articles on efficient and optimal economic growth in the Review of Economic Studies, Vol. 34, 1967).

Recall that with a finite dimensional commodity space, say R^ℓ, a price system (p_i) is simply an ℓ-vector such that if $y = (y_i)$ is any consumption program involving these commodities, the value of y relative to p is given by

$$p(y) = \sum_{i=1}^{\ell} p_i\, y_i \qquad (1.1)$$

From the expression (1.1) the following useful properties are clear:

a. The value is well-defined for any y in R^ℓ and any p in R^ℓ;

b. For fixed p it is a homogenous linear function of y, and vice versa;
c. It is continuous in p and y.

By virtue of (a), it is possible to compare the values of any two programs; and a price being associated with each commodity it is possible to talk about ratios of prices, which may be used to measure the rates of exchange between commodities.

In Section 2 we shall present several examples of situations involving infinitely many commodities, and show in each case that the set of all technologically possible consumption programs (measured in suitable units) can be represented as a proper subset of the linear space $L_{\infty}(\Omega, \mathcal{F}, \mu)$, where $(\Omega, \mathcal{F}, \mu)$ is some suitably chosen complete; σ −ddefinite measure space; and may be interpreted as the set of "commodities".

By assigning the μ − essential sup-norm to L_{∞}, and considering a price system or a value function to be a linear functional continuous in the norm topology, it is clearly possible to conserve (a) and (b) and continuity in y; if, further, the space of all continuous linear functions (L_{∞}^{*}) is endowed with the weak topology, one gets joint continuity. It can be shown that if Y is convex set representing all possible consumption programs andy^{*} is an efficient point of Y (i.e., maximal in the vector ordering), then y^{*} is indeed value-maximizing relative to some non-zero p^{*}, i.e., there exists a non-zero p^{*} in (L_{∞}^{*}) with $p^{*}(y^{*}) \geq p^{*}(y)$ for all y in Y.

In most application to economic theory, if p^{*} is to have a useful economic interpretation as a price system, then it must be represented in the form

$$p^{*}(y) = \int_{\Omega} g^{*} y du \qquad (1.2)$$

Where g^{*} is a μ −integratable functions on the set of commodities. Unfortunately, this representation is not possible in general. A theorem of Yosida and Hewitt (2002) shows that a general representation of p^{*}, i.e.,

$$p^{*}(y) = \int_{\Omega} g^{*} y du + \int_{\Omega} y du \qquad (1.3)$$

Where g^{*} is as in (1.2), and v is a bounded pur… finitely additive measure that vanishes on sets o… − measure zero (see Section 3). In problems… dynamic optimization with an infinite ti… horizon, the second term on the right side of (… reflects the asymptotic behaviour of the progr… y, whereas in problems of decision in the fac… uncertainty, this term can be interpreted … reflecting the behavior of program y on sets… "arbitrarily small probability". It is, of course, … first part of (1.3) that corresponds most directly… (1.1), since $g^{*}(\omega)$ can be interpreted as the "pr… of commodity ω".

Two approaches can be taken to deal with t… problems introduced by the second term in (1.… first, one can give various compactness conditi… under which every maximal program in t… feasible set can be approximated by maxim… programs for which the associated price system… of the form (1.2) (Section 4). Second, one ca… give conditions (Section 5) that guarantee that o… can choose a price system of the from (1.2), ev… though there may be other price systems in whi… the second part of (1.3) is not zero. No profits w… be included in the present paper; for the le… obvious proofs the reader will be referred to iter… in the bibliography at the end.

Economic Examples

In this section we present several econom… models, in order to motivate the more formal a… abstract presentation that follows in t… subsequent sections, as well as to illustrate t… diversity of models whose analysis can be unifie… by the functional analysis approach.

Example 1. A Discrete Time Infinite Horiz… Model

Some of the typical difficulties involved … problems of inter-temporal resource allocati… appear even in a simple aggregative model wi… just one producible good that can be used either … consumption or as an input for further producti… of the same good (or it can be thrown away- "fr… disposal"). Let q_t denote the output at date t, … the capital stock, I_t the (gross) investment, and … the consumption (= 1, 2, ….). The initial cap… stock, k_0, is given. The output at date …

determined by the capital stock at the previous date:

$$q_t = f(k_{t-1}) \qquad (2.1)$$

The capital stock is assumed to depreciate at a constant rate, δ, which is strictly between 0 and 1; thus, if $\sigma = 1 - \delta$,

$$k_{t+1} = \sigma k_t + I_t \qquad (2.2)$$

By convention, consumption plus investment cannot exceed output, and the capital stock must be nonnegative:

$$y_t + I_t \le q_t, \quad k_t \ge 0. \qquad (2.3)$$

The nonnegative function f is called the production function. A negative investment may be interpreted as "eating up" or "running down" the capital stock. A negative consumption implies a need to borrow or obtain "aid" from outside the system. Given the initial capital stock, k_0, the set Y of feasible consumption sequences, (y_t), is determined by (2.1) – (2.3) for $t \ge 0$. A consumption sequence $y^* = (y_t^*)$ in Y is efficient or maximal in Y if there is no other sequence y in Y such that for all t, $y_t \ge y_t^*$, and for some t, $y_t > y_t^*$. If U is a real – valued function on Y, then $\hat{y}$ is called optimal in Y with respect to U if it maximizes $U(y)$in Y.

Let p_t denote the price of the (single) commodity at date t. The total value of consumption is (assuming no disposal)

$$\Sigma p_t y_t = \Sigma p_t[f(k_{t-1}) - k_t + \sigma k_{t-1}] \qquad (2.4)$$

$$= p_1[f(k_0) + \sigma k_0] + \Sigma_{t\ge 2}(p_{t+1}[f(k_t) + \sigma k_t] - p_t k_t)$$

A consumption sequence $y^* = (y_t^*)$ maximizes the value (2.1) in the set Y if, and only if, for each $t \ge 1$, the corresponding capital stock, k_t^*, maximizes

$$p_{t+1}[f(k_t) + \sigma k_t] - p_t k_t, \qquad (2.5)$$

Or, if $p_t > 0$, the expression

$$[1/(1 + r_t)]\,[f(k_t) + \sigma k_t] - k_t, \qquad (2.6)$$

Where the ratio $(1 + r_t) = (p_t/p_{t+1})$ is called the interest factor at date t, and r_t is the corresponding interest rate.

Since k_t is the input of capital at date t, and $[f(k_t) + \sigma k_t]$ is resulting new output plus the surviving capital, the expression (2.6) may be interpreted as the "inter-temporal profit" at date t, where the value of new output plus surviving capital stock is "discounted" according to the interest rate r_t. Alternatively, expression (2.5) may be interpreted as "inter-temporal profit at date t discounted from date 0". Thus, if an efficient program can be characterized as maximizing the value (2.4) for some sequence (p_t), then it can equivalently be characterized as maximizing inter-temporal profit at each date, provided the sum (2.4) converges. Notice that if the sequences in Y are uniformly bounded, and (p_t) is absolutely summable, then the sum (2.4) converges for every program in Y. However, discounted inter-temporal profit (2.5) is well defined at each date for any program (y_t) and any sequence (p_t), whether or not the total value (2.4) converges.

Example 2. A Continuous Time Model

Even with a finite horizon, the treatment of time as a continuous variable leads to a continuous of commodities. As an illustration, we present now a version of the previous example, with continuous time and finite horizon. We also introduce the idea of a primary resource, which is necessary for production but is not produced and is supplied exogenously. Consider the time interval [0, T], where 0 is the present instant, and T > 0 is the "horizon". At time t, let output, capital stock, labour input, and investment be denoted by $Q(t), K(t), L(t), and\ I(t)$, respectively; output, labour and investment are flows, whereas capital is a stock. Corresponding to (2.1) and (2.2) of Example 1, we have

$$Q(t) = F[K(t), L(t)], \qquad (2.7)$$

$$K(t) = -\sigma K(t) + I(t), K(t) \ge 0, \qquad (2.8)$$

Assume that the labour input is determined exogenously, by

$$L(t) = L(0)e^{\lambda t}, and\ L(t) > 0, \qquad (2.9)$$

And that the production function, F, is homogenous of degree one ("constant returns to scale"). With these two assumptions we can reduce the variables of the system to per capita terms. Define

$$q(t) = \frac{Q(t)}{L(t)}, k(t) = \frac{K(t)}{L(t)}, y(t) = \frac{[Q(t)-I(t)]}{L(t)}, f(x) = F(x,1) \quad (2.10)$$

Then, corresponding to (2.3), (2.7), and (2.8), one can verify that

$$q(t) = f[k(t)], \quad (2.11)$$

$$y(t) + k(t) \leq q(t) - (\delta + \lambda)k(t), k(t) \geq 0 \quad (2.12)$$

If we further impose the condition that per capita consumption, $y(t)$, be nonnegative, then (2.11) and (2.12) reduce to

$$k(t) \leq f[k(t)] - (\delta + \lambda)k(t), k(t) \geq 0, \quad (2.13a)$$

$$y(t) = f[k(t)] - (\delta + \lambda)k(t) - k(t) \quad (2.13b)$$

We also impose initial and terminal conditions on the capital stock:

$$k(0) = k^*, k(T) = k_* \quad (2.13c)$$

Where k^* is the given initial ratio of capital stock to labour input, and k_* is the terminal value of $k(t)$ that one wants to attain.

Define the set π of feasible capital accumulation programs as the set of nonnegative absolutely continuous functions k on [0, T] satisfying (2.13a – c), and let Y be the set of corresponding per capita consumption programs defined by (2.13b).

Example 3. A Multi- Sector Model

A natural extension of the one – producible-good model discussed in the first two examples is one in which there is a finite number of goods (say ℓ) in which there is a finite number of goods (say ℓ) at each date. Again, taking time as a discrete variable, assume that at each date t (= 1, 2 ...), for every nonnegative input vector in R^{ℓ}, there is a set $p_t(a)$ of possible output vectors available at date (t + 1) for possible consumption or for use as an input in production. The correspondence P_t is the production possibility corresponding at date t. The graph G_t of P_t is the set $\{(a,b): b \text{ is in } P_t(\ldots$ in $R^{2\ell}$.

A production program is a sequen... $\{(a_t, b_{t+1}): t = 1, 2, \ldots\}$ such that $a_t \geq 0$, ... $b_{t+1} \geq 0$, and b_{t+1} is in $P_t(a_t)$. For ... production program, the net output program defined by

$$\left.\begin{array}{l} y_1 = -a_1 \\ y_t = b_t - a_t, t \geq 2 \end{array}\right\} \quad (2.1\ldots)$$

The set of all net output programs $y = (y\ldots$ corresponding to all possible production progra... is denoted by Y. one may wish to impose ... additional constraint, corresponding to th... constraint that consumption be nonnegative vect... w_t of quantities of commodities exogenous... supplied to the economy. Since consumption a... equals $(y_t + w_t)$, we may impose th... constraint$(y_t + w_t) \geq 0$; the correspondi... subset of Y will be denoted by Y_w.

A particular case of the foregoing is the line... activity analysis model. For each t let A_t and B_t be given nonnegative matrices, respectively, an... let the production correspondence P_t be defin... by

$$P_t(a_t) = \{b_{t+1}: \text{ for some } x_t \geq 0, \; b_{t+1} \leq B_t x\ldots \; a_t \geq A_t X_t\}.$$

The vector X_t is called the vector of activity leve... at date t. for a more general discussion of suc... multisector models, see Radner (2007) an... references given there, especially Malinvau... (2003).

Example 4. Allocation under Uncertainty

Finally, we present an example of dynam... resources allocation under uncertainty. In th... example, the technology and resources at ea... date depend upon the history of the (stochasti... environment up to that date. The model presente... here is an extension of the model of Example3.

Let S_t denote the finite set of all events that c... occur at date t, and let S^t denote the set of t... tuples, $S^t = (s_1, \ldots, s_t)$, with s_t in S_t. The t... tuples, s^t is a possible history of the environme... from date 1 through date t. Suppose that (s...

$s_2, \ldots, ad\ inf.$) is a stochastic process, i.e., there is a probability measure on $S = X_{t=1}^{\infty} S_t$, and that at each date the conditional distribution of future events, given the past events and economic decisions, does not depend on the economic decisions taken. In other words, the environment is stochastic and exogenous (this is by now a standard approach in decision theory and economics; see Savage, 2004, and Ch. 7 of Debreu, 2009).

The production correspondence of Example 3 is generalized as follows: If $a_t$is the vector of inputs at date t, then $P_t(a_t, s^t)$ is a set of "stochastic outputs", i.e., a set of functions from S^{t+1} to R_+^{ℓ}, the nonnegative orthant of R^{ℓ}. A production program is a sequence (α_t, β_t) such that (i) α_t *and* β_t are functions from S^t to R_+^{ℓ}, and (ii) for every s^t in S^t, $\beta_{t+1}$is in $P_t[a_t(s^t), s^t]$. If we write $a_t = \alpha_t(s^t)$ and $b_t = \beta_t(s^t)$, then the (stochastic) net outputs, y_t, are determined by (2.14), and we may write $y_t = n_t(s^t)$. The set Y of net output program is therefore the set of sequences (n_t) corresponding to the set of production programs. If the exogenous supplies of resources, w_t (see Example 3), are also stochastic, i.e., determined by corresponding functions v_t on S^t, and we constraint consumption to be nonnegative, then, corresponding to the set Y_w of Example 3 we get the set Y_w, where $v = (v_t)$. Models along the lines of the present example have been used in Radner (2001a, 2001b), JeanJean (2001), and Stigum (2001).

Programs and Prices

Let $(\Omega, \mathcal{F}, \mu)$ be a complete, definite measure space. In our models, Ω can be identified with the set of all commodities. Consider$L_\infty(\Omega, \mathcal{F}, \mu)$, the linear space of all real – valued measurable functions that are bounded μ –almost everywhere. For y in $L_\infty(\Omega, \mathcal{F}, \mu)$ one can define the "μ –essup norm" as

$$\|y\| = \inf_N \sup_{\omega \in \Omega \sim N} |y(\omega)|, \quad (3.1)$$

Where N ranges over sets of μ –measure zero. A special example is obtained by taking Ω to be any countable set (in this case, without loss of generality Ω can be chosen to be the set of all positive integers), $\mathcal{F}$ the σ –field of all subsets of Ω and μ the counting measures; in such a case, L_∞ is simply the space of all bounded real sequences (known as ℓ_∞). At the end of this section we shall indicate how for each of the examples in the previous section the set Y of feasible consumption programs can be identified with a convex subset of L_∞, provided certain assumptions are made and the units of measurement are suitably chosen. Since point y in L_∞ specifies the quantity $y(\omega)$ of every commodity ω, we shall call points of L_∞ programs.

A program y in L_∞ is nonnegative, written $y \geq 0$, if $y(\omega) \geq 0$ for almost every (a. s) ω in Ω, is positive, written $y > 0$, if $y \geq 0$ and $\mu(\omega: y(\omega) > 0$. For y_1, y_2 in L_∞, $y_1 \geq y_2$ (resp $y_1 > y_2$) if $y_1 - y_2 \geq 0$ (resp. $y_1 - y_2 > 0$). For a subset Y of L_∞, y^* in Y is a maximal or efficient element of Y if there does not exist another y in Y, with $y = y^*$.

A price system or a value functional is simply a linear functional on L_∞ that is continuous in the topology generated by the norm defined in (3.1). The set of all value functional is denoted by L_∞^*. For p in L_∞^*, define

$$\|p\|_* = \lim_{\|y\| \leq 1} |p(y)| \quad (3.2)$$

For p in L_∞^*, p is nonnegative (written $p \geq 0$) if $p(y) \geq 0$ for all $y \geq 0$; p is positive ($p > 0$) if $p \geq 0$ and $p \neq 0$, and p is strictly positive ($p \gg 0$) if $p \geq 0$ and $p \neq 0$; and is strictly positive ($p \gg 0$) and $p(y) > 0$, for all $y \geq 0$. Since the measure space $(\Omega, \mathcal{F}, \mu)$ is assumed to be complete and σ –finite, if p is in L_∞^*, one has the following representation (Yosida and Hewitt, 2002)

$$p(y) = \int_\Omega ygdu + \int_\Omega ydv_p \quad (3.3)$$

Where g is an integratable function on Ω (i.e., g belongs to $L_1(\Omega, \mathcal{F}, \mu)$ and v_p is a bounded, purely finitely additive signed measure that vanishes on all sets of μ –measure zero, g will be called the "L_1 –part of p" and v_p the "purely finitely additive part of p", if $v_p = 0$, we shall say that "p is in L_1" when p is in L_1, we have

$$p(y) = \int_\Omega ygdu$$

so that the value of the program y relative to p [i.e., the number p(y)] is an integral over the set of commodities g(ω) is the unit price of commodity ω, and the situation is analogous to that of the special case in which Ω is finite and value is the scalar product of the price and quantity vectors. In the case of ℓ_∞, we have, for any $y = (y_\omega)$ in ℓ_∞ and p in L^*_∞ .

$$p(y) = \sum_{\omega=1}^{\infty} g_\omega y_\omega + \int_\Omega y dv_p \qquad (3.4)$$

Where g_ω is in ℓ_1.

The existence of the purely finitely additive part creates problems of economic interpretation. It should be mentioned at the outset that even for a nonzero $p \geq 0$, it is possible for the L_1 – part of p to be zero. However, if p is represented by a purely finitely additive measure, then only a "small" set of commodities can possibly have nonzero value, as the following lemma shows:

Lemma 3.1 (Yosida and Hewitt, 2002):

If Ψ is a countable additive measure on $(\Omega,\mathcal{F})$ with Ψ(Ω) finite, and v is any purely finitely additive measure, then for any $\epsilon > 0$, there is A_ϵ in $\mathcal{F}$ with $\Psi(A_\epsilon) < \epsilon$ and $v(\Omega \sim A_\epsilon)=0$. Going back to the example of a countable Ω- and thus without loss of generality Ω may be chosen to be the set of positive integers ("dates" ω =1, 2, ...,). The Yosida – Hewitt lemma implies that if the ℓ_1 –part of p is zero, the value of any plan $y = (y_\omega)$ converges to $\bar{y}$ (as $\omega \to \infty$), then for any p (with $\|p\|_* = 1$) that is represented by a purely finitely additive measure, $p(y) = \bar{y}$; more generally one can show that

$$\lim\inf y_\omega \leq p(y) \leq \lim\sup y_\omega \qquad (3.5)$$

Whenever p has a zero ℓ_1 –part and $\|p\|_* = 1$ (in Lemma 3.1, let Ψ assign the measure ($\frac{1}{2^\omega}$) to pint ω in Ω). If Y is a subset of L_∞, y^* is a program in Y, p is a price system in L^*_∞, and

$$p(y) \leq p(y^*) \text{ for all y in Y.} \qquad (3.6)$$

The we shall say that y^* is value maximizing (in Y) with respect to p, or equivalently that p supports Y at y^*. The following theorems relate the concepts of efficiency and value maximization (see Majumder, 2001a).

Theorem 3.1:

If p in L^*_∞ support a subset Y of L_∞ at y^* in Y, and $p \gg 0$, then y^* is maximal in Y. on the other hand, if Y is convex, and y^* is maximal in Y, then there is a $p > 0$ in L^*_∞ that supports Y at y^*. Examples of programs supported by a purely finitely additive price system are given in Radner (2007), McFadden (2008), and Peleg – Yeari (2000).

We now reconsider the examples of Section 2. In each case we shall show that the set Y of feasible consumption programs has certain compactness properties. The relevance of these compactness properties will become apparent in the next section:

Example 1. Take Ω to be the nonnegative integers (the dates), and μ to be the counting measure. Assume that the production function, f, is nonnegative and concave, that f(0) = 0, and that $f'(k)$ approaches 0 as k increases without limit. Under this assumption, Y is convex, and one can show (see Koopmans, 2005) that fir every k_t there is a constant c such that $y_t \leq c$ for every feasible consumption program. Hence the set Y_+ of nonnegative feasible consumption programs is a convex norm – bounded subset of ℓ_∞. One can also show that Y_+ is closed in the product topology, and therefore in the $w(\ell_\infty, \ell_1)$ topology. Hence Y_+ is a convex, $w(\ell_\infty, \ell_1)$—compact subset of ℓ_∞ (see Dunford and Schwarz, 2007, V. 4, 3).

Example 2. Take Ω to be the interval [0, T], $\mathcal{F}$ to be the Borel sets of {0, T}, and μ to be Labesque measure. Assume that F(0, L) = F(k, 0) = 0 (i.e., that both capital and labour are necessary in production), that F is strictly concave and homogenous of degree one, and that f is twice differentiable with $f'(0) > \delta + \lambda$. Koopmans (2005) has shown that under these assumptions π is a norm – bounded subset of $\mathcal{C}[0, T]$, considered as a subset of L_∞. From this one can show that π is a norm – compact convex subset of $\mathcal{C}[0, T]$, and that Y is a $w(\ell_\infty, \ell_1)$—compact convex subset of L_∞ (see Majumdar, 2001b).

Example 3. We shall say that a production possibility correspondence Pm is neoclassical if (i) P(0) = (0), (ii) the graph of P is closed convex cone, and (iii) b in P(a), $b' \leq b, a' \geq a$, imply b' in P(a') ('free disposal"). One can show (see Radner, 2007) that if at every date t, P_t is neoclassical, than for any given sequence (w_t) of exogenous resources there is a consequence (h_t) of nonnegative numbers such that, for all consumption programs (y_t) in Y_w, $\|y_t\| \leq h_t$ for t (where $\| \quad \|$denotes the Euclidean norm in R^ℓ). It follows from this that there is a change of units at each date such that Y_w can be represented as a convex, norm- compact subset of ℓ_∞.

Example 4. Since at each date t there are only finitely many partial histories $s^t = (s_1, ..., s_t)$, and since inputs and outputs at each date t are functions of s^t, the analysis of this example is similar to that of Example 3, and will not be given here (see Majumdar, 2000b). The situations is more complicated if we wish to study "stationary" stochastic economies, in which case it is convenient to think of the history of the environment as extending infinitely far back into the past. (we may imaging that we are observing the economy after it has been operating under a given policy for a long time.) in this case one is led to represent Y as a subset of L_∞, where Ω is the set of doubly infinite sequences (s_t), $\mathcal{F}$ is the sigma – field generated by cylinder acts, and μ is a stationary probability measure on Ω, i.e., a measure invariant under the shift transformation $(Ts)_t = s_{t+1}$ (see Radner, 2001b).

Approximation of Maximal Programs by Regular Maximal Programs

A maximal program y^* in Y will be called regular if Y is supported at y^* by some nonnegative, non zero price system in L_1, i.e., if the supporting price system has no purely finitely additive part. In this section we give conditions under which maximal programs can be approximated by regular maximal programs. Our first theorem is taken from Majumdar (2001a).

Theorem 4.1. suppose Y is a convex $w(L_\infty, L_1)$— compact subset of L_∞: then the $w(L_\infty, L_1)$— closure of the set of regular maximal points of Y contains the set of maximal points of Y. Since the weak [or, $w(L_\infty, L_\infty^*)$] topology, by using Eberlen's theorem (see Dunford – Schwartz, 2007, V. 6. 1) and the proof of theorem 4.1, one can get the following:

Theorem 4.2: let Y be a convex, $w(L_\infty, L_\infty^*)$ - compact subset of L_∞: then the $w(L_\infty, L_\infty^*)$ – sequential closure of the set of regular maximal points of Y contains the set of maximal points of Y. when the measures space (Ω, $\mathcal{F}$, μ) is finite, one can substitute "$w(L_\infty, L_1)$- closure" in the statement of Theorem 4.1 (imitate the first part of the proof Dunford – Schwartz, 2007, V. 6. 1).

For a subset Y of L_∞, $w(L_\infty, L_1)$-topology (see Dunford – Schwarz, 2007, V. 4. 3). Relatively mild restrictions (like continuity of the production function) ensure closeness. Roundedness is guaranteed by a suitable choice of units when the exogenous resources impose a restriction on the rate of growth of the economy.

Weak convergence is to be interpreted as convergence in value; thus, $(y^d, d \in D)$ converges to y^* in the $w(L_\infty, L_1)$-topology if the valuesp(y^d) converges to p(y) for every value functional p in L_1. A sharpened version of convergence—convergence of programs, prices and values – can proved if Y is assumed to be convex and compact in the norm topology of L_∞. The latter undoubtedly is a strong restriction. In what follows L_∞^* is endowed with the $w(L_\infty, L_\infty^*)$ or weak topology; any product of topological spaces is, of course, assigned the product topology.

Theorem 4. 3: let Y be a convex, norm-compact subset of L_∞ and y^*be a maximal point of Y; then

(i) There is $p^* > 0$ in L_∞^*, $\|p^*\|_* = 1$, supporting Y at y^*;

(ii) (y^*, p^*) is the limit of a net p^d (y^d) such that for each $d \in D$,

(ii.a) y^d is a maximal point of Y,

(ii.b) p^d>0, $\|p^d\|_* = 1$ in L_1, and p^d supports Y at y^d.

Thus, y^* is the norm-limit of a net $\{y^d : d \in D\}$ of regular maximal points: the value functional p^d

associated with the approximating y^d are themselves convergent to some p^* relative to which y^* is value maximizing. Of course, p^d (y^d) converges to p^* (y^*), and $\{y^d: d \in D\}$ converges to y^* in $w(L_\infty, L_\infty^*)$- topology (see the counterexample in Kothe, 2009), p.311). if Y is a convex subset of ℓ_∞ that is compact in the product topology, any maximal point y^* is the limit of a sequence $y^n = (y_t^n)$ of maximal points such that for each n there is a strictly positive $p^n = (p_t^n)$ $(p_t^n > 0 \; for \; all \; t)$, with $\sum_{t=1}^{\infty} y_t^n p_t^n \geq \sum_{t=1}^{\infty} p_t^n y_t^n)$ for all y in Y. hence, if Y is compact in any stronger topology, the price systems associated with the approximating regular maximal points can be chosen to be strictly positive. Also, note that, in ℓ_∞ convergence in the $w(\ell_\infty, \ell_1)$- topology implies point wise convergence; hence, theorem 4.1 when applied to ℓ_∞ also yields the result that any maximal point y^* is the limit in the topology of point wise convergence of a net of regular maximal points (in addition to convergence in value relative to each p in ℓ_∞). A detailed discussion of the approximation question for subsets in ℓ_∞ is to be found in Majumdar (2000a) and Peleg (2001).

Conditions under which a Maximal Program is Regularly Maximal

The theorems of Section 4 indicate that maximal programs are regular, i.e., are value-maximizing with respect to a price system in L_1, except in "limiting" cases. The present section gives two situations in which it can be directly verified that a maximal program is regular. In the first situation, the maximal program also maximizes a function on L_∞ that has suitable continuity, concavity, and monotonicity properties. Such a function may be interpreted as the "utility" or "objective" function. A real-valued function U on $L_\infty$is called quasi-concave at $\bar{y}$ if {y in L_∞: U(y) $\geq U(\bar{y})$} is convex. We shall say that U is weakly – monotone if $x \geq y$ implies $U(x) \geq U(y)$ and $x \gg y$ implies $U(x) > U(y)$ [where $x \geq y$ means that $x(\omega) > y(\omega)$; further, U is strongly-monotone if $x > y$ implies $U(x) > U(y)$. Denote by $t(L_\infty, L_1)$ the Mackey topology on L_∞ relative to L_1, i.e., the strongest topology on L_∞ such that the points of L_1 are continuous on L_∞ (see Kothe, 2006).

Theorem 5.1: Let y^* be a maximal point of convex set Y in L_∞. If there exists a real – value function U such that

(a). $(y^*) \geq U(y)$ for all y in Y,

(b). U is quasi – concave at y^*,

(c). U is weakly – monotone and is continuous the $t(L_\infty, L_1)$ - topology of L_∞, then there exists nonzero p^* in L_1, $p^* > 0$, such that $p^*(y^*)$ ≥ $p^*(y)$ for all y in Y. If U satisfies the strong monotonicity condition, then p^* is strict positive, i.e., $p^* \gg 0$. (For a proof of theorem 5 see Majumdar, 2001a).

Our second theorem is applicable to a situation in which the set Y satisfies a certain "mixture" condition. In this situation, if a program is value maximizing with respect to a price system, then is value- maximizing with respect to the L_1 –part of the price system (see section 3). We shall say that a set Y in L_∞ has the "mixture" property if for every y' and y'' in Y, and every set A in $\mathcal{F}$, the program y is also in Y, where y is defined by $y(\omega) = y'(\omega)$ for ω in A and $y(\omega) = y''(\omega)$ for ω not in A.

Theorem 5.2: If Y has the mixture property; and y^* is value – maximizing with respect to p, then is also value – maximizing with respect to the L_1 part of p.

Corollary: If Y has the mixture property, if y^* maximal in Y, if p^* supports Y at y^*, if p^* is not purely finitely additive, then y^* is regular. (For proof, see Majundar, 2001a) This result, and related techniques, have been applied in Bewley (2000) and Radner (2001b).

Efficiency and Inter-temporal Profit Maximization

In this Section we reconsider the multi – sector model (Example 3), and briefly indicate how by applying Theorem 4.3, one can derive a system of prices relative to which efficient programs can be characterized in terms of inter-temporal profit maximization. First, for simplicity, let us assume that there are primary resources, and that the units of measurement are, in fact, chosen in such a way that Y_w as well as all the sequences (a_t), (b_{t+1})

inputs and outputs generating the elements of Y_w are in ℓ_∞ (see Radner, 2007). A production program (a_t, b_{t+1}) is inter-temporally profit maximizing with respect to a sequence(p_t) of vectors in R^ℓ if for every $t \geq 1$

$$p_{t+1}b_{t+1} - p_t a_t \leq p_{t+1}\hat{b}_{t+1} - p_t\hat{b}_{t+1} \text{ for all } (a_t, b_{t+1}) \text{ in } G_t. \quad (6.1)$$

The following theorem relates value maximization to inter-temporal profit maximization (for a proof, see Radner 2007, and Majumdar and Kurz, (2000).

Theorem 6.1: Suppose p^* in ℓ_∞^* supports Y_w in ℓ_∞ at y^*=(y_∞^*). If (a_∞^*, b_{t+1}^*) is the production program generating y^*=(y_t^*), then (a_∞^*, b_{t+1}^*) is inter-temporally profit maximizing, and y^*=(y_t^*) is value maximizing relative to the ℓ_1- part of p^*. Thus, if the ℓ_1- part of the value functional supporting an efficient program in nonzero, one can simply disregard the purely finitely additive part of p^*. From (3.5) it follows that if y^*=(y_t^*) is such that there is another $y = (y_t)$ in Y_w with $lim\, inf\, y_t > lim\, sup\, y_t^*$, then the ℓ_1- part cannot be zero. Even when the ℓ_1- part of the supporting functional p^* is zero, provided that the efficient program y^*=(y_t^*), satisfies a particular technological condition, it is possible to obtain a nonzero sequence (p_t) of prices relative to which the corresponding production program is inter-temporally profit maximizing. Actually, Theorem 6.1 can be stated in a way such that it is applicable even if there are no primary resources (see Radner, 2007). We shall state the next theorem in full generality, making it directly comparable to the original version of Malinvaund (2003). We shall use a few definitions.

A good is non producible at date t if the correspondence coordinate of the output vector b is zero for every (a, b) in G_t; otherwise, it is producible. Suppose that the (possibly empty) set of non producible goods is the same for each t; inputs, a, and outputs, b, are represented in the form

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_2 \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ 0 \end{pmatrix},$$

(6.2)

Where a_1 , a_2 are the inputs of producible and non producible goods, respectively, and b_1 is the vector of producible goods. A program ($a_t, b_{t+1}; y_t$) is called tight at date: if there is no (a', b') in G_t such that

$$a_2' \ll (a_2)_t,$$
(6.3)

$$b_1' \gg (b_1)_{t+1}$$

Where "≪ " means each coordinator of a_2 is strictly greater than the corresponding coordinator of a_2', etc, of course the first line of (6.3) applies only if there are non producible goods. The proof of the following theorem is in Radner (2007).

Theorem 6.2: Suppose

(i) For each t, G_t is closed, convex and contains (0, 0),

(ii) λ_t is a given sequence of positive real numbers converging to zero.

(iii) $\|\hat{y}_t\| \leq \lambda_t, \|\hat{a}_t\|\lambda_t, \quad \|\hat{b}_{t+1}\| \leq \lambda_{t+1},$ for all $t \geq 1$,

(iv) $(\hat{a}_t, \hat{b}_{t+1}; \hat{y}_t)$ is a feasible program such that $(\hat{y}_t)$ is efficient in Y ∩ ℓ_∞,

(v) $(\hat{a}_t, \hat{b}_{t+1}; \hat{y}_t)$ is not tight at any date t; then there exists a sequence $(\bar{p}_t)$ of non negative vectors in R^ℓ, not all zero, such that $(\hat{a}_t, \hat{b}_{t+1})$ is inter temporally profit maximizing with respect to $(\bar{p}_t)$

CONCLUSION

It should be noted that, under condition (i), if $(\hat{a}_t, \hat{b}_{t+1}; \hat{y}_t)$ is any feasible program, then units of measurement can always be chosen so that condition (iii) is satisfied. In addition, if there are primary resources, units can be chosen so that for all programs in Y_w, (iii) will be satisfied (see Radner, 2007, for details). An example of tight program such that there is no nonzero system of price relative to which it is profit maximization relative to a strictly positive system of prices is not sufficient for efficiency (see Malinvaud, 2003).

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